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Weak normality of subsets of exp (X)

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Abstract

If X is a countably compact space and if $\exp(X)$ is hereditarily weakly normal, then X is a perfectly normal hereditarily separable compact space. All powers of a T_1-space X are weakly normal if and only if X is compact T_2. © 1997 Elsevier Science B.V.

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M.M. Čoban [4] proved that if $\exp(X)$ is hereditarily normal, then X is a metrizable compact space. Here the exponential space $\exp(X)$ is the set of all nonempty closed subsets of X with Vietoris (finite) topology [5, 2.7.20]. A T_1-space X is said to be *weakly normal* [1,2], if for every two disjoint closed subsets A and B of X there exists a continuous mapping f of X into $\mathbb{R}^\omega$ such that images of A and B are disjoint. Clearly, every normal space is weakly normal. If there exists a one-to-one continuous mapping of X onto a separable metrizable space, then the space X is weakly normal. So the Niemytzkij plane or the square of the Sorgenfrey line can serve as examples of weakly normal spaces which are not normal [1]. Can one use weak normality instead of normality in Čoban's theorem? It is easy to see that this is not the case. Indeed, there exists a one-to-one continuous mapping of $\exp(\omega)$ onto D^ω, so $\exp(\omega)$ is hereditary weakly normal, but the discrete countable space ω is not compact.

Theorem 1. *If X is a countably compact space and if $\exp(X)$ is hereditarily weakly normal, then X is a perfectly normal hereditarily separable compact space.*

Further we need the following lemmas.

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Lemma 1 [11]. *$(\omega_1 \times (\omega + 1)) \setminus (LIM \times \{\omega\})$ is not weakly normal.*

Lemma 2. *Let F and H be closed disjoint subsets of a normal space X such that H is infinite and $\exp(X)$ is hereditarily weakly normal. Then F is a separable G_δ-set in X.*

Proof. Let $\Phi_0 = \{x_0\} \subset F$, $\Phi_{\xi+1} = \Phi_\xi \cup \{x_{\xi+1}\}$, where $x_{\xi+1} \in F \setminus \Phi_\xi$ and

$$\Phi_\xi = \overline{\bigcup\{\Phi_\lambda\colon \lambda < \xi\}},$$

if ξ is a limit number. We have $\Phi_0 \subset \cdots \subset \Phi_\xi \subset \Phi_{\xi+1} \subset \cdots \subset \Phi_{\omega_1}$, if F is not separable. It is easy to see that the subspace $\{\Phi_\xi \in \exp(F)\colon \xi \leqslant \omega_1\}$ is homeomorphic to $\omega_1 + 1$ (see [6]). And $\omega + 1 \subset \exp(H)$, because H is infinite. So $(\omega_1 + 1) \times (\omega + 1) \subset \exp(X)$. This is a contradiction with Lemma 1. Thus F is separable. There exists an open set $U \supset F$, such that $\overline{U} \cap H = \emptyset$, because X is normal. Then $\exp(H) \times \exp(\overline{U})$ is a subset of $\exp(X)$ and hence this product is hereditarily weakly normal. Again $\omega + 1 \subset \exp(H)$. If F is not a G_δ-set in X, then F is not a G_δ-set in $\overline{U}$ too. Let v be a system of open subsets of U such that $|v| = \tau$ and $\bigcap v = F$. Let us suppose that τ is minimal with the property that there is such a system. Clearly, $\tau \geqslant \omega_1$, because F is not a G_δ-set in $\overline{U}$. Let $v = \{V_\xi\colon \xi < \tau\}$ and $G_\xi = \bigcap\{\overline{V}_\lambda\colon \lambda \leqslant \xi\}$. If $\alpha_0 = 0$ and $\alpha_{\xi+1}$ is the first number, for which $G_{\alpha_\xi} \setminus G_{\alpha_{\xi+1}} \neq \emptyset$ and $\alpha_\xi = \sup\{\alpha_\lambda\colon \lambda < \xi\}$ for a limit number ξ, then $\sup \xi = \tau$, and we can assume that

$$G_0 \supset G_1 \supset \cdots \supset G_\lambda \supset G_{\lambda+1} \supset \cdots \supset G_\xi \supset \cdots \supset G_\tau = F.$$

Again $(\omega_1 + 1) \subseteq B = \{G_\xi \in \exp(\overline{U})\colon \xi \leqslant \tau\}$ [6]. Thus $(\omega_1 + 1) \times (\omega + 1) \subset \exp(X)$, and this contradiction with Lemma 1 proves Lemma 2. □

Proof of Theorem 1. Suppose that X is not compact and let ϕ be a centered system of closed sets such that $|\varphi| = \tau$ and $\bigcap \varphi = \emptyset$. Let us suppose that τ is minimal with the property that there is such a system. Clearly, $\tau \geqslant \omega_1$, because X is countably compact. Let $\phi = \{F_\xi\colon \xi < \tau\}$ and $G_\xi = \bigcap\{F_\lambda\colon \lambda \leqslant \xi\}$. Let us pick up a nonisolated point $x \in X$ (this is possible, because X is noncompact, but countably compact). There exists $\beta < \tau$ such that $x \notin G_\beta$, because $\bigcap\{G_\xi\colon \xi < \tau\} = \emptyset$. Let H be a closed neighborhood of x such that $H \cap G_\beta = \emptyset$. (Let us note here that X is normal; see Lemma 3 below.) Let $\alpha_0 = \beta$ and $\alpha_{\xi+1}$ be the first number, for which $G_{\alpha_\xi} \setminus G_{\alpha_{\xi+1}} \neq \emptyset$ and $\alpha_\xi = \sup\{\alpha_\lambda\colon \lambda < \xi\}$ for a limit number ξ. It is evident that $\sup \xi = \tau$, so we can assume that $G_0 \supset G_1 \supset \cdots \supset G_\lambda \supset G_{\lambda+1} \supset \cdots \supset G_\xi \supset \cdots$ and $H \cap G_0 = \emptyset$. Then $\omega_1 \subseteq B = \{G_\xi\colon \xi < \tau\}$ and B is a closed subset of $\exp(G_0)$ (see [6]). Since H has a non-isolated point x, $\omega + 1 \subset \exp(H)$. So $(\omega + 1) \times \omega_1 \subset \exp(H) \times \exp(G_0)$, and $\exp(H) \times \exp(G_0) \subset \exp(X)$, because $H \cap G_0 = \emptyset$. But $(\omega + 1) \times \omega_1$ contains a subspace, which is not weakly normal (Lemma 1). Thus X is a compact space.

If X contains two nonisolated points $x \neq y$, one can choose open disjoint sets U and V such that $x \in U$ and $y \in V$. Then, if F is a closed subset of X, then $F = K \cup L$, where $K = F \setminus U$ and $L = F \setminus V$. Lemma 2 implies that K and L are separable G_δ-sets in X and hence F is a separable G_δ-set in X. We see that the pseudocharacter of X

is countable and X is compact, so X is first-countable. Hence every subset of X is separable [10, Proposition 3]. If X contains only one nonisolated point x, then X is the one-point compactification of the discrete space $X \setminus \{x\}$. It is easy to see in this case that $X \times (\omega + 1) \subset \exp(X)$ and this space is not hereditarily weakly normal, if X is not countable (see also Lemma 2 of [7]). □

Corollary. *If X is countably compact and* $\exp(\exp(X))$ *or* $\exp(X \times X)$ *is hereditarily weakly normal, then X is a metrizable compact space.*

Proof. We see that in this case $\exp(X)$ and $X \times X$ are perfectly normal compact spaces. This implies metrizability of X.

Problem. Is a compact space X metrizable, if $\exp(X)$ is hereditarily weakly normal?

Let us note that sometimes one can use weak normality instead of normality. For example, Yakivchik [11] gave a generalization of the Tamano theorem [5, 5.1.38]. The paper [8] contains a generalization of the theorem from [3]: if X is compact and $X^2 \setminus \Delta$ is normal, then X is first-countable. Arhangel'skii proved the next fundamental fact.

Lemma 3 [1]. *Every weakly normal countably compact space is normal.*

This fact allows us to give the next generalization of Noble's theorem [9]: all powers of a T_1-space X are normal if and only if X is compact T_2.

Theorem 2. *All powers of a T_1-space X are weakly normal if and only if X is compact T_2.*

Proof. It is sufficient to prove that all powers of X are countably compact and to use Lemma 3 after that. It is easy to see that if some power contains ω as a closed subset, then there exists τ such that ω^{ω_1} is a closed subset of X^τ. And it is sufficient to prove that ω^{ω_1} is not weakly normal. Let A_k, $k = 1, 2$, be the set of all points $\{x_\alpha\colon \alpha < \omega_1\} \in \omega^{\omega_1}$ such that for each integer n other than k, $x_\alpha = n$ for at most one $\alpha < \omega_1$. The sets A_k are closed and disjoint [5, 2.7.16(a)]. Thus if ω^{ω_1} were weakly normal there would exist continuous $f : \omega^{\omega_1} \to \mathbb{R}^\omega$ such that $f(A_1) \cap f(A_2) = \emptyset$. Since f depends on countably many coordinates [5, 2.7.12(c)] there exist $\beta < \omega_1$ and $f_0 : \omega^\beta \to \mathbb{R}^\omega$ such that $f = f_0 \circ p_\beta$ and $p_\beta : \omega^{\omega_1} \to \omega^\beta$ is the projection. Let us fix a one-to-one correspondence $\zeta : \omega \setminus \{1, 2\} \to \beta$, and let us define $x \in A_1$ and $y \in A_2$ such that $x_\alpha = y_\alpha = \zeta(\alpha)$ if $\alpha < \beta$, $x_\alpha = 1$ and $y_\alpha = 2$ if $\alpha \geqslant \beta$. Then $p_\beta(x) = p_\beta(y)$ and $f(x) = f(y)$. This contradiction proves Theorem 2. □

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